"SHALLOW WATER" APPROXIMATION FOR VORTEX FLOW OF A PERFECT FLUID

V. Yu. Lyapidevskii

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Introduction. This paper models a system of equations describing planar unsteady motion of a perfect incompressible fluid in a channel. For a specific class of flows, e.g., those with a symmetric velocity profile, the equations are reduced to a one-dimensional unsteady system. It should be noted that, depending on the solution, the nonlinear system of equations obtained can be both elliptic and hyperbolic. The question arises as to the correctness of the Cauchy problem for the equations considered [1].

It is shown that the flow corresponding to the elliptic case is unstable towards small perturbations. Stability is taken to mean stability of the linearized system in established flow with straight streamlines. The development of a Kelvin–Helmholtz instability leads to mixing of the fluid layers and transition to a stable velocity profile.

On the basis of this mixing mechanism the author suggests a model for the potential motion of two fluid layers separated by a vortex region. The choice of velocity profile over the channel cross section is based on minimizing the energy loss in transition to stable flow, and is accomplished by using the conservation laws for mass and momentum. Flow of fluid over a step is examined as an example of the possible unique rearrangement of the velocity profile. Allowance for shear instability prevents elliptic regions appearing in the solutions considered, and makes it necessary to study the Cauchy problem for hyperbolic systems of equations with constraints.

1. We consider plane-parallel flow of an ideal incompressible fluid in a channel bounded above and below by horizontal impermeable walls. The fluid fills the entire channel, and therefore the motion is determined by the initial velocity distribution. The flow is described by the system of Euler equations

$$\begin{aligned} &u_t + uu_x + wu_y + p^*_x = 0, && -\infty < x < \infty, \\ &w_t + uw_x + ww_y + p^*_y = 0, && -H < y < H, \\ &u_x + w_y = 0, && t > 0, \end{aligned} \tag{1.1}$$

where u and w are the coordinates of the velocity vector; $p^* = p + gy$ is the "modified" pressure, and g is the acceleration due to gravity. At the boundaries we have the impermeability conditions

$$w|_{y=-H} = w|_{y=H} = 0. \tag{1.2}$$

The "shallow water" approximation has been widely used to describe "smooth" flows in which the parameters of the motion change appreciably only at distances considerably greater than the channel width. This approximation consists in eliminating from the Euler equations the terms describing the vertical acceleration, which leads to a hydrostatic pressure distribution. The equations can be obtained formally with the aid of the following expansion of the dependent and the independent variables in the system (1.1) [2]:

$$x \to x,\ y \to \varepsilon y,\ t \to \varepsilon^{-1/2} t,\ u \to \varepsilon u,\ v \to \varepsilon^{3/2} v,\ p \to \varepsilon p.$$

Retaining terms of first order in ϵ in Eq. (1.1), we have

$$\begin{aligned} &u_t + uu_x + wu_y + p^*_x = 0, \\ &u_x + w_y = 0,\ p^*_y = 0. \end{aligned} \tag{1.3}$$

Without loss of generality we may assume that H = 1. The impermeability conditions, Eq. (1.2), remain unchanged: $w|_{y=-1} = w|_{y=1} = 0$. It follows from Eq. (1.3) that the pressure is independent of y, i.e., $p^* = p^*(t, x)$, and the function $\omega = u_y$ satisfies the equation

$$\omega_t + u\omega_x + w\omega_y = 0. \tag{1.4}$$

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Thus, the quantity ω is conserved along trajectories of the solution of Eq. (1.3), analogously to the nonzero component of vorticity in solutions of the system (1.1). We shall use this property to simplify the system (1.3).

We consider a continuous solution of Eq. (1.3) with lines of weak discontinuity $y = h \pm \eta$, where $\eta \geqslant 0$, and h and η are smooth functions of t and x. Let $\omega(t, x, y) = 0$ outside the layer $h - \eta \leqslant y \leqslant h + \eta$. This corresponds to the flow being potential in the vicinity of the walls $y = \pm 1$. We note that $u = u(t, x)$ in the potential flow region. For the mean velocity and the intensity of the vortex layer we introduce the following notation: $v = {}^1/_2(u|_{y=h-\eta} + u|_{y=h+\eta})$, $\gamma = {}^1/_2(u|_{y=h+\eta} - u|_{y=h-\eta})$. Because of Eq. (1.3) the quantities v, γ, h, η satisfy the equation

$$\begin{gathered}\gamma_t + (\gamma v)_x = 0,\\ \omega^+[(1 - h - \eta)_t + ((1 - h - \eta)(v + \gamma))_x] = 0,\\ \omega^-[(1 + h - \eta)_t + ((1 + h - \eta)(v - \gamma))_x] = 0,\end{gathered} \tag{1.5}$$

where $\omega^\pm$ are the discontinuities in the function ω on the lines $y = h \pm \eta$. The first equation is the difference in the momenta for the functions $u = u|_{y=\pm 1}$. The other two equations come from integrating the continuity equation at the weak discontinuity and at the walls. To close the system (1.5) we introduce the mean vorticity $\bar{\omega} = \gamma/\eta$. Using the condition that the mass flow over the channel section is constant,

$$\int_{-1}^{1} u\,(t, x, y)\,dy = q\,(t),$$

we obtain the equation

$$\bar{\omega}_t + v\bar{\omega}_x + \frac{\bar{\omega}}{\eta}\left(\eta v - \frac{1}{2}\int_{h-\eta}^{h+\eta} u dy\right)_x = 0.$$

If the instantaneous velocity profile satisfies the condition

$$v = \frac{1}{2\eta}\int_{h-\eta}^{h+\eta} u dy, \tag{1.6}$$

we add to the system (1.5) the equation

$$\bar{\omega}_t + v\bar{\omega}_x = 0. \tag{1.7}$$

In particular, if $\omega \equiv$ const for $|y - h| < \eta$, then Eq. (1.6) is satisfied and $\omega = \bar{\omega}$. We note that, because of Eq. (1.4), it is sufficient to require that $\omega \equiv$ const for $t = 0$, and this property will hold for all $t > 0$.

We consider the system (1.5) and (1.7) with $\omega(t, x) = \bar{\omega}(t, x)$. Then $q(t) = 2(v - \gamma h)$, and the system takes the form

$$\begin{gathered}\eta_t + ({}^1/_2\eta q + \omega\eta^2 h)_x = 0,\\ h_t + ({}^1/_2 hq + \omega\eta(\eta + h^2 - 1))_x = 0, \quad \omega_t + v\omega_x = 0.\end{gathered} \tag{1.8}$$

The lines

$$dx/dt = \lambda_i(t, x), \quad i = 0, 1, 2$$

are characteristics of the system, where $\lambda_0 = v$, $\lambda_{1,2} = v + \gamma h \pm \gamma\sqrt{2\eta + h^2 - 1}$. The condition for the system (1.8) to be hyperbolic is expressed by the inequality

$$\eta \geqslant {}^1/_2(1 - h^2). \tag{1.9}$$

A reasonable problem for Eq. (1.8) is the Cauchy problem, and thus one should investigate the correctness of the Cauchy problem for a nonlinear system of equations which varies its type, depending on the solution.

A way out of this difficult position is noted below. It will be shown that the inequality (1.9) describes stability of a flow with an appropriate velocity profile towards small perturbations. The development of a shear flow instability when condition (1.9) breaks down is modelled by transition to a stable profile, using the laws of conservation of mass and total momentum and the Bernouilli constants. With this approach the solution always remains in the hyperbolic region. Naturally, the question of solubility of the Cauchy problem for the system of equations (1.8) with bounds of this type also requires a thorough investigation.

Note 1. The system (1.8) in the hyperbolic region is reminiscent of the equations of gas dynamics. Vorticity plays the role of entropy. If the vorticity is constant at the initial time, the system converges to two solutions, but only until shock waves are formed or the constraint (1.9) is applied.

2. To determine the solution in the hyperbolic region one must use the conservation laws which define the laws of discontinuity in the solution. If we consider solution discontinuities only of small amplitude, then we can determine the general solution from the following divergent form of the system (1.5), (1.7):

$$(1+h-\eta)_t + ((1+h-\eta)(v-\gamma))_x = 0,$$
$$(1-h-\eta)_t + ((1-h-\eta)(v+\gamma))_x = 0, \tag{2.1}$$
$$\eta_t + (\eta v)_x = 0,\ \gamma_t + (\gamma v)_x = 0.$$

Let v be the speed of propagation of a line of discontinuity x = x(t) in the solution of the system (2.1). On the discontinuity line we have the Hugoniot conditions

$$\nu[h-\eta] = [(1+h-\eta)(v-\gamma)],\ \nu[h+\eta] = ([h+\eta-1)(v+\gamma)],\ \nu[\gamma] = [\gamma v],\ \nu[\eta] = [\eta v]. \tag{2.2}$$

The discontinuity lines in the solution of the system (2.1) correspond to a mathematical description of internal hydraulic jumps, i.e, the flow region with a sharp variation in parameters, for which the shallow water approximation examined in Section 1 is not suitable.

We now convert to a coordinate system moving with velocity $\nu = dx/dt = \text{const}$, that of the discontinuity line. Let the flow on both sides of the discontinuity have piecewise linear velocity profiles $(u_0(y), 0)$, $(u(y), 0)$. The quantities h, η, v, γ, characteristics of the profiles u(y), $u_0(y)$, are related by Eq. (2.2) ($\nu = 0$ in this coordinate system). These relations state that the Bernoulli constants are conserved in regions of vortex-free motion, and that the total mass flux and the energy are conserved across a discontinuity line. In addition, it follows from Eq. (2.2) that the vorticity $[\omega] = 0$ and the fluid mass are constant in each of the layers $-1 \leqslant y \leqslant h-\eta$, $h-\eta \leqslant y \leqslant h+\eta$, $h+\eta \leqslant y \leqslant 1$, i.e., there is no mixing of layers in passing through a hydraulic jump. However, total momentum is not conserved, and therefore Eq. (2.2) can be regarded as a convenient approximation describing a discontinuity of moderate intensity. The analogous approach in gasdynamics is replacing the energy conservation law by the condition that entropy is constant in passing through a discontinuity line.

3. We turn now to the question of stability of a flow with parallel streamlines and its connection with condition (1.9). The functions u = u(y), w = 0, p* = const are particular solutions of the problem of Eqs. (1.1), (1.2). However, an established flow with velocity profile u = u(y), w = 0 may be unstable towards small unsteady perturbations. The development of a shear instability, called a Kelvin-Helmholtz instability, leads to strong mixing of the fluid layers. The result is a new stable velocity profile. Here there is no explicit account for the influence of viscosity, which causes energy loss in the fluid in forming vortices.

We consider possible methods of passing from an unstable velocity profile to a stable one, while conserving integral flow characteristics such as mass flux and momentum. An unstable velocity profile may arise, for example, because of the boundary conditions. For instance, when two uniform streams with different velocities flow together, the velocity profile is strongly unstable. This leads to a breakdown in the contact surface and the formation of some monotonic velocity profile at quite a large distance from the beginning of intermixing.

We limit the examination to class Λ stable profiles u(y) with the following properties:

a) the function u(y) is twice continuously differentiable in the intercept [−1, 1];

b) the function u(y) increases (or decreases) monotonically;

c) there exists a unique point of inflection $\bar{y}$

$$\bar{y} \in (-1, 1),\ u'(\bar{y}) = \max_{y\in(-1,1)} u'(y) \left(u'(\bar{y}) = \min_{y\in(-1,1)} u'(y)\right);$$

d) the inequality

$$\Delta = \frac{1}{u'(y)\left(u(y)-u(\bar{y})\right)}\Bigg|_{-1}^{1} + \int_{-1}^{1} \frac{u''(y)\,dy}{(u'(y))^2\left(u(y)-u(\bar{y})\right)} > 0 \tag{3.1}$$

holds. Condition "d" is the necessary and sufficient condition for stability of the profile u(y) with properties "a"–"c" towards unsteady harmonic perturbations of arbitrary wavelength [3].

We shall consider that in some coordinate system moving at constant velocity along the channel, the process of transition from an unstable profile to a stable one has been accomplished. We shall neglect the influence of the boundary layer near the rigid walls. Therefore for y = ±1 it follows from the Bernouilli integral that the quantity $1/2\ u^2 + p^* = \text{const}$. Let $u_0(y)$ be an unstable profile, undergoing transition to $u(y) \in \Lambda$. Then the relations

$$\int_{-1}^{1} u(y)\,dy = \int_{-1}^{1} u_0(y)\,dy, \int_{-1}^{1} (u^2(y) + p^*)\,dy = \int_{-1}^{1} (u_0^2(y) + p_0^*)\,dy,$$
$$\frac{1}{2}u^2(1) + p^* = \frac{1}{2}u_0^2(1) + p_0^*, \frac{1}{2}u^2(-1) + p^* = \frac{1}{2}u_0^2(-1) + p_0^* \tag{3.2}$$

are valid. We now can try to solve the problem of finding the function u(y) of class Λ so as to minimize the energy jump $[E] = E_0 - E$, where

$$E = \int_{-1}^{1} \left(\frac{1}{2}u^3(y) + p^*u(y)\right) dy$$

subject to conditions (3.2) and limits (3.1). This is a problem of classical calculus of variations. It is not difficult to show that an extremum of the functional is not reached in class Λ. We then go to a wider class $\bar{\Lambda}$ which includes the limits of functions from Λ. By convergence we understand uniform convergence in the interval $[-1, 1]$ and convergence to C^2 in some interval containing the point of inflection of the sequence considered. In class $\bar{\Lambda}$ functions suspected of an extremum will be continuous piecewise linear functions, consisting of two parts $u \equiv$ const (extremals in Λ) and a straight line section joining them (one-sided extremum). We denote this class by Λ_0. We shall not try to show that a minimum of the functional [E] in class $\bar{\Lambda}$ is reached in elements of Λ_0. The rationale for introducing class Λ is, first, that it contains functions most frequently used for approximating the velocity distributions in the mixing layer, e.g., the hyperbolic tangent, and, secondly, the necessary and sufficient condition (3.1) for flow stability is known for a velocity profile from this class. Therefore, we shall regard the foregoing discussion as being directed at a choice of class Λ_0, and shall now discuss the possibility of using condition (3.2) to determine the profile u(y) of class Λ_0.

Lemma 1. For a function u(y) of class Λ_0, the inequality

$$(b - a)^2 - 4(1 + a)(1 - b) \geqslant 0 \tag{3.3}$$

is satisfied, where the points $y = a$, $y = b$ are inflection points of the piecewise linear function $u(y)$ $(-1 \leqslant a \leqslant b \leqslant 1)$.

Proof. The function u(y) is a limit of stable profiles $u_n(y)$ of class Λ in the following sense:

A) $\sup\limits_{y\in(-1,1)} |u(y) - u_n(y)| \to 0, \quad n \to \infty;$

B) there exists $\bar{y} \in (-1,1)$, $\bar{y} = \lim\limits_{n\to\infty} \bar{y}_n$, where $u''(\bar{y}_n) = 0$;

C) there exists a quantity $\delta > 0$ such that $u_n(y) \to u(y)$ for $n \to \infty$ in the space $C^2_{[\bar{y}-\delta,\bar{y}+\delta]}$.

It follows from properties A-C that $\bar{y} \in (a, b)$ for $u(a) \neq u(b)$. We note that expression $\Delta(u_n)$ can be written in the form

$$\Delta(u_n) = \frac{1}{u_n'(u_n - \bar{u}_n)}\Bigg|_{\bar{y}_n-\delta}^{\bar{y}_n+\delta} + \int_{\bar{y}_n-\delta}^{\bar{y}_n+\delta} \frac{u_n''dz}{(u_n')^2(u_n - \bar{u}_n)} - \int_{-1}^{\bar{y}_n-\delta} \frac{dz}{(u_n - \bar{u}_n)^2} - \int_{\bar{y}_n+\delta}^{1} \frac{dz}{(u_n - \bar{u}_n)^2} > 0,$$

where $\bar{u}_n = u_n(\bar{y}_n)$. Going to the limit in the expression $\Delta(u_n)$, we obtain the inequality

$$\Delta(u) = \frac{1}{u'(u - \bar{u})}\Bigg|_{\bar{y}-\delta}^{\bar{y}+\delta} - \int_{-1}^{\bar{y}-\delta} \frac{dz}{(u - \bar{u})^2} - \int_{\bar{y}+\delta}^{1} \frac{dz}{(u - \bar{u})^2} \geqslant 0.$$

We note that

$$u'(z) = \begin{cases} 0, & -1 \leqslant y < a, \\ \omega, & a < y < b, \\ 0, & b < y \leqslant 1. \end{cases}$$

Therefore, we can write the expression for $\Delta(u)$ in the form

$$\Delta(u) = \frac{1}{\omega^2}\left[\frac{1}{b - \bar{y}} - \frac{1}{a - \bar{y}} - \frac{1 + a}{(\bar{y} - a)^2} - \frac{1 - b}{(\bar{y} - b)^2}\right].$$

Since $\Delta(u) \geqslant 0$, we come to the inequality

$$-2\bar{y}^2 + (2a - a^2 + 2b + b^2)\bar{y} + 2a^2b - a^2 - 2ab^2 - b^2 \geqslant 0,$$

i.e., $d^2 = (a - b)^2((a - b)^2 - 4(a + 1)(1 - b)) \geqslant 0$, which was to be proved.

Lemma 2. Flow with a profile from Λ_0 is stable.

Proof. We can directly investigate the stability of a flow with a velocity profile (u(y), 0) for $u(y) \in \Lambda_0$. Because of the piecewise linearity of the function u(y) the Rayleigh equations take a simple form [4]. We seek a solution in the form of a combination of exponents. Here the coefficients are determined by the conditions for continuity of the solution and by the impermeability conditions at $y = \pm 1$. Bounded growth of perturbations with time for arbitrary wavelength occurs only when inequality (3.3) is fulfilled. This condition was derived in [4] in the special case for $a = -b$.

We now turn to determination of a u profile from Λ_0 for a given profile u_0. For convenience we consider that the u_0 profile is constructed in the same way as the u profile, as long as it satisfies the stability condition (3.3). Let $h = {}^1/_2(a + b)$, $\eta = {}^1/_2(b - a)$, $v = {}^1/_2(u(-1) + u(1))$, $\gamma = {}^1/_2(u(1) - u(-1))$.

Note 2. The stability condition (3.3) coincides with the hyperbolic condition (1.9) of system (1.8), derived from the assumption that the instantaneous velocity profile is piecewise linear.

For the functions u and u_0 of class Λ_0 relations (3.2) have the form

$$\begin{gathered} v - \gamma h = v_0 - \gamma_0 h_0, \quad \gamma v = \gamma_0 v_0, \\ v^2 - 4\gamma h v + \gamma^2\left(1 - \frac{4}{3}\eta\right) = v_0^2 - 4\gamma_0 h_0 v_0 + \gamma_0^2\left(1 - \frac{4}{3}\eta_0\right). \end{gathered} \tag{3.4}$$

To close the system (3.4) we add the stability condition (3.3) for the u(y) profile

$$\eta = {}^1/_2(1 - h^2). \tag{3.5}$$

We note than an equality sign has been used in condition (3.5), since, when a stable state is reached, there is no further mixing and Eqs. (2.2) come into force. We now proceed to analyze the relations obtained.

Lemma 3. For a stable profile $u_0(y)$ Eqs. (3.4) and (3.5) have solutions with $v(v_0 - v) \geqslant 0$.

Proof. Let $\gamma_0 v_0 \neq 0$. Then $v \neq 0$ and the equation in v has the form

$$P(v) = -7v^4 + 8v^3(v_0 - \gamma_0 h_0) - v^2\left(v_0^2 - 8\gamma_0 h_0 v_0 + \gamma_0^2(3 - 4\eta_0 - 2h_0^2)\right) + \gamma_0^2 v_0^2 = 0. \tag{3.6}$$

Since the polynomial P(v) has different signs at the point $v = 0$ and $v = v_0$: $P(0) = \gamma_0^2 v_0^2 > 0$, $P(v_0) = 2(2\eta_0 - 1 + h_0^2)\cdot \gamma_0^2 v_0^2 < 0$, then we have proved that there is a root v^* of the polynomial P(v). The other relations can be solved uniquely. For the case $\gamma_0 v_0 = 0$ a solution can be found in explicit form. Thus the lemma has been proved.

Note 3. It follows from Eq. (3.6) that there is at least one more root v_*, where $v^* v_* \leqslant 0$. From the assumption that in the vicinity of the walls the streamlines are directly uniquely along the x axis, it follows that the inequality

$$(v \pm \gamma)(v_0 \pm \gamma_0) \geqslant 0 \tag{3.7}$$

holds. From Eqs. (3.4) and (3.7) we obtain the conditions $vv_0 \geqslant 0$, $\gamma\gamma_0 \geqslant 0$, which eliminate the root v_* of Eq. (3.6). However, two real roots of the equation $P(v) = 0$ may appear. In addition, the system (3.4), (3.5) is nonlinear and its solution satisfies certain constraints, e.g., $-1 \leqslant h \leqslant 1$, and we must still check that these are satisfied.

4. We now investigate a solution of the system (3.4), (3.5) in detail in the example of flow of fluid over a step (see Fig. 1). We consider established flow of the fluid in the region $x > 0$, $-1 < y < 1$. Uniform flow is given at the section $x = 0$, $y \in (h_0, 1)$. The impermeability condition holds at the channel walls. The problem is to determine the velocity profile $u(y) \in \Lambda_0$ at sufficiently large values of $x > 0$, using Eqs. (3.4), (3.5). The initial velocity profile $u_0(y)$ at $x = 0$ is piecewise constant ($\eta_0 = 0$)

$$u_0(y) = \begin{cases} 0, & -1 < y < h_0, \\ u_0, & h_0 < y < 1. \end{cases}$$

Therefore, $v_0 = \gamma_0 = {}^1/_2 u_0 > 0$. The system (3.4), (3.5) takes the form

$$\begin{gathered} v - \gamma h = v_0(1 - h_0), \quad \gamma v = v_0^2, \\ v^2 - 4\gamma v h + \gamma^2\left(1 - \frac{4}{3}\eta\right) = 2v_0^2(1 - 2h_0), \quad 2\eta = 1 - h^2. \end{gathered} \tag{4.1}$$

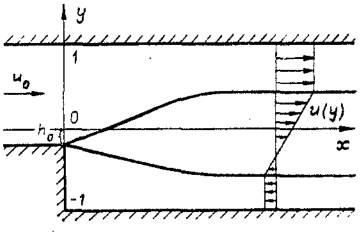

Fig. 1

Lemma 4. There exists a unique solution of system (4.1), satisfying condition (3.7).

Proof. Let $v = \alpha v_0$. We note that only positive values of α satisfy condition (3.7). In fact, $v + \gamma = v_0(\alpha + \alpha^{-1}) > 0$ only for $\alpha > 0$. We shall show that the desired value $\alpha^* \in [0, 1]$. In this case the constraint $|h| = |\alpha^*(\alpha^* - 1 + h_0)| < 1$ is satisfied. From Eq. (4.1) we obtain an equation in α, accounting for η by the parameter ($\eta = {}^1/_2(1 - h^2)$, i.e., $0 \leqslant \eta \leqslant 1/2$):

$$Q(\alpha) = -3\alpha^4 + 4(1 - h_0)\alpha^3 - 2(1 - 2h_0)\alpha^2 + 1 - \frac{4}{3}\eta = 0. \tag{4.2}$$

Since $Q(0) = 1 - {}^4/_3\eta > 0$, $Q(1) = -{}^4/_3\eta \leqslant 0$, then evidently the root α^* of the polynomial $Q(\alpha)$ exists in the interval $[0, 1]$. This value of α^* gives the solution of system (4.1).

We shall now prove the uniqueness of the positive root in Eq. (4.2). The extrema of the function $Q(\alpha)$ are located at the points $\alpha_0 = 0$ and

$$\alpha^{\pm} = \frac{3(1 - h_0) \pm \sqrt{3(3h_0^2 + 2h_0 - 1)}}{6}. \tag{4.3}$$

Therefore, for $-1 \leqslant h_0 \leqslant 1/3$ the discriminant in Eq. (4.3) is negative and for $\alpha > 0$ the function $Q(\alpha)$ is monotonic. For $h_0 \in [{}^1/_2, 1]$ at the point $\alpha = 0$ there is a local minimum, and a maximum is reached in the interval $[0, 1]$. Therefore, the root of Eq. (4.2) for $\alpha > 0$ is unique. It remains to consider the case $h_0 \in ({}^1/_3, {}^1/_2)$. Here $0 < \alpha^- < \alpha^+ < 1$. A minimum is reached at the point α^-, and at the point α^+ the function $Q(\alpha)$ has a maximum. Since $\partial Q/\partial h_0 = 4\alpha^2(1 - \alpha) > 0$ for $\alpha \in (0,1)$, the function $Q(\alpha^-(h_0))$ increases monotonically for $h_0 \in ({}^1/_3, {}^1/_2)$ and reaches a minimum at the point $h_0 = 1/3$. Here $Q\left(\alpha^-\left(\frac{1}{3}\right)\right) = Q\left(\frac{1}{3}\right)\Big|_{h_0=1/3} = 1 - \frac{4}{3}\eta - 3^{-4} > 0$ for $0 \leqslant \eta \leqslant 1/2$. The lemma has been proved.

We note that the problem examined in Section 4 arises in the study of flow of an ideal fluid over a network of profiles of finite thickness.

The proposed model is an attempt to describe mathematically the development of a shear instability in the motion of an ideal fluid in the long-wave approximation. The stability condition (1.9) leads to the system of equations obtained being hyperbolic and gives reason to hope that the Cauchy problem for this system is correctly posed.

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